Formula for the phase velocity of electromagnetic waves

Shihua Huang,* Fengmin Wu, and Bo Hu

Department of Physics, Zhejiang Normal University, Zhejiang 321004, China

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A new formula for the phase velocity of electromagnetic waves presented by Chen *et al.* [Appl. Phys. Lett. **88**, 121125 (2006)] is investigated and discussed here. The difference between the result obtained with the new formula and that obtained directly using the phase term is small for a fundament-mode Gaussian laser beam. However, this difference is qualitative in some high-order Gaussian-mode laser beams. Using the new formula for such beams, discontinuities arise in the distribution of the phase velocity. This distribution is not rotationally symmetric with respect to the optical axis, and an imaginary phase velocity may appear near these discontinuities.

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I. INTRODUCTION

Much research on the measurement or discussion of velocity (phase velocity, group velocity, etc.) has been reported in the literature, e.g., [1]. Phase and phase velocity are important physical variables for describing wave fields [2,3]. Velocity is always a topic of interest because of its role in some long-standing problems [4,5] and its close relationship with various applications [6–8]. In a recently published paper [9], an exact expression for the phase velocity has been derived for a monochromatic wave field in a homogeneous medium on the basis of the fundamental wave equation. This expression is significant in that the phase velocity is sufficiently determined by the wave amplitude with no explicit reference to the phase. Thus, it is possible to obtain the phase velocity distribution without measuring the phase. In fact, measuring the phase motion directly is a challenging experimental technique [10]. The objective of this paper is to investigate the new formula given in Ref. [9] and to calculate the phase velocity of a Gaussian laser beam with different modes and compare the results to that obtained using the phase terms directly. For simplicity, throughout this paper, length is normalized by $1/k=\lambda/2\pi$ (λ is the wavelength) and time by $1/\omega$ (ω is the angular frequency of the optical wave); then the wave vector value is $k=1$ and the speed of light $c=1$.

II. FORMULAE OF THE PHASE VELOCITY

In the original paper [9], the wave function is assumed to be of the form $\psi(\mathbf{r},t)=\psi_{\mathbf{r}}(\mathbf{r})e^{i\varphi(\mathbf{r},t)}$, where $\psi_{\mathbf{r}}(\mathbf{r})$ and $\varphi(\mathbf{r},t)$ are real functions and $\psi_{\mathbf{r}}(\mathbf{r})$ is the wave amplitude. This means that $\psi_{\mathbf{r}}(\mathbf{r})$ is always positive. However, in practical situations, $\psi_{\mathbf{r}}(\mathbf{r})$ may be negative, for example, for high-order Laguerre–Gaussian (LG) and Hermite–Gaussian (HG) laser beams. In fact, if $\psi_{\mathbf{r}}(\mathbf{r})$ is not always positive, their deduction is also correct. Accordingly, we should assume that $|\psi_{\mathbf{r}}(\mathbf{r})|$ is the wave amplitude and the optical-field intensity is similar as $I=|\psi_{\mathbf{r}}(\mathbf{r})|^2$. An important conclusion of the original paper [9] is that one can obtain the phase velocity distribution by measuring the field intensity. Even if $\psi_{\mathbf{r}}(\mathbf{r})$ is not always positive, this conclusion is also right because we can get

$$\frac{\nabla^2\psi_{\mathbf{r}}}{\psi_{\mathbf{r}}}=\frac{1}{2}\left(\frac{\nabla^2 I}{I}-\frac{1}{2}\frac{(\nabla I)^2}{I^2}\right) \tag{1}$$

and from the new formula in Ref. [9]

$$v_p=c\left[1+\frac{\nabla^2\psi_{\mathbf{r}}}{k^2\psi_{\mathbf{r}}}\right]^{-1/2}. \tag{2}$$

We know that the phase velocity v_p is still a function of the laser field intensity I.

HG and LG mode laser beams are particular solutions of the paraxial equation, which is an approximation of the *Helmholtz* equation, under the slowly varying-envelope approximation (SVEA) [11]. So it is reasonable that the above formula deduced from the *Helmholtz* equation is valid under the SVEA for HG- and LG-mode laser beams. For an x-polarized HG(m,n)-mode laser beam, its transverse electric component E_x can be expressed as [12]

$$E_{mn}=\frac{a_{mn}w_0}{w(z)}H_m(X)H_n(Y)\exp\left(-\frac{r^2}{w(z)^2}\right)\times\exp\left\{-i\left[(m+n+1)\phi(z)+\phi_0-\frac{kr^2}{2R(z)}\right]\right\}\times\exp[ik(z-ct)]. \tag{3}$$

Here a_{mn} is the reference strength, w_0 is the beam width at focus, $X=\sqrt{2}x/w(z)$, $Y=\sqrt{2}y/w(z)$, $H_m(X)$ and $H_n(Y)$ are Hermite polynomials, and

$$\phi(z)=\tan^{-1}(z/Z_R), \tag{4}$$

$$R(z)=z(1+Z_R^2/z^2), \tag{5}$$

$$Z_R=kw_0^2/2, \tag{6}$$

$$w(z)=w_0(1+z^2/Z_R^2)^{1/2}, \tag{7}$$

$$r^2=x^2+y^2. \tag{8}$$

Then, the phase of E_x is given by

*Corresponding author; hsh@zjnu.cn

$$\varphi = k(z-ct) - (m+n+1)\phi(z) - \phi_0 + \frac{kr^2}{2R(z)}. \quad (9)$$

Similarly to Ref. [11], the value of the phase velocity can be obtained by

$$v_p = ck/|\nabla\varphi| = ck\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]^{-1/2}. \quad (10)$$

Namely,

$$v_p = ck\left\{\frac{2fz^2}{Z_R^2 w(z)^2} + \left[k - \frac{(m+n+1) - f + r^2/w_0^2}{kw(z)^2/2}\right]^2\right\}^{-1/2}, \quad (11)$$

where

$$f = \frac{2r^2}{w(z)^2}. \quad (12)$$

To distinguish from the new formula, we will call Eq. (11) as the old formula.

For convenience, if we just take the x-axial high-order modes into account, namely, $n=0$, we have

$$\psi_{\mathbf{r}}(\mathbf{r}) = \frac{a_m w_0}{w(z)} H_m(X) \exp\left(-\frac{r^2}{w(z)^2}\right). \quad (13)$$

Then, we can get

$$\frac{1}{\psi_{\mathbf{r}}}\frac{\partial^2\psi_{\mathbf{r}}}{\partial x^2} + \frac{1}{\psi_{\mathbf{r}}}\frac{\partial^2\psi_{\mathbf{r}}}{\partial y^2} = \frac{2f - 4(m+1)}{w(z)^2}, \quad (14)$$

$$\frac{1}{\psi_{\mathbf{r}}}\frac{\partial^2\psi_{\mathbf{r}}}{\partial z^2} = (z^2 + Z_R^2)^{-2}\Bigg\{[(f^2 - 2mf - 5f + 4m + 2)z^2 + (f - m - 1)Z_R^2] + \left[(X^2 - f + 2)z^2 - \frac{Z_R^2}{2}\right] \times \left[2(2X^2 - 2m - 1) - \frac{H_{m+2}(X)}{H_m(X)}\right]\Bigg\}. \quad (15)$$

Then, substituting Eqs. (14) and (15) into the new formula [Eq. (2)], we can get the phase velocity distribution of the HG$(m,0)$-mode laser beam.

III. DIFFERENCE BETWEEN THE TWO APPROACHES

A. Fundamental-mode case

If the electric field is written as $E = \tilde{E}e^{-ikz}$, the conditions for the SVEA can be expressed as [13]

$$\left|\frac{\partial^2\tilde{E}}{\partial z^2}\right| \ll \left|2k\frac{\partial\tilde{E}}{\partial z}\right| \quad \text{or} \quad \left|\frac{\partial^2\tilde{E}}{\partial x^2}\right| \quad \text{or} \quad \left|\frac{\partial^2\tilde{E}}{\partial y^2}\right|. \quad (16)$$

We know these scales as $\nabla_\perp\tilde{E} \sim 1/kw_0$, $\partial\tilde{E}/\partial z \sim 1/Z_R$, and $\partial^2\tilde{E}/\partial z^2 \sim 1/Z_R^2$ [14]. All these mean that the bigger the w_0 is, the smaller the difference between the paraxial equation and the *Helmholtz* equation will be. Accordingly, the difference between the result obtained with the new formula (corresponding to solutions satisfying the *Helmholtz* equation) and that obtained with the old formula for a Gaussian-mode laser beam (corresponding to solutions satisfying the paraxial equation) will decrease as w_0 increases. This is consistent with the conclusion in the original paper [9]. Similarly as Ref. [9], we find that the lowest-order approximate expressions of the two approaches are exactly the same for a fundamental-mode Gaussian laser beam. In the focus plane ($z=0$), the contribution from the first term inside the brace of Eq. (11) is zero. In the xz plane, the contribution from some of the other terms cannot be counted in. So we select points in the line ($y=w_0/2$, $z=w_0$) for analysis. Figure 1(a) shows the phase velocity as a function of x along this line [solid lines from using Eqs. (2), (14), and (15), dotted lines from using Eq. (11)] in continuous fundamental-mode laser beams with different beam widths $kw_0=60,30,10,6$. We find that their differences increase as w_0 decreases. If $w_0 \to \infty$, their difference is zero because for a plane wave, $v_p=c$, and the contribution from the term $\nabla^2\psi_{\mathbf{r}}/k^2\psi_{\mathbf{r}}$ is zero.

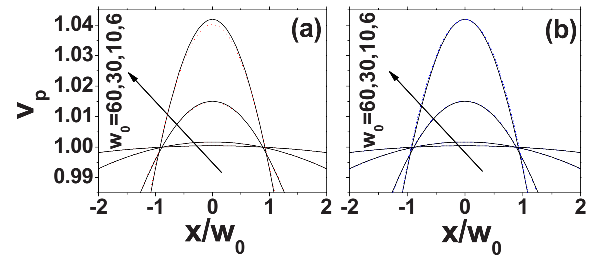

FIG. 1. (Color online) The phase velocity as a function of x along the line ($y=w_0/2$, $z=w_0$). Continuous fundamental-mode laser beams with different beam widths $k_0w_0=60,30,10,6$ are used. (a) Using the new formula (solid lines) and the old formula (dotted lines) with a paraxial approximation solution. (b) Using new formula and the field intensity obtained with a seventh-order correction solution (dash-dotted lines) and the same as (a) for solid lines.

Nowadays, a laser beam can be focused onto several wavelengths, even up to its diffraction limit. To describe this kind of tightly focused laser field accurately, many high-order correction solutions beyond the paraxial approximation solutions have been proposed [15]. The expressions for these high-order correction solutions are so complicated that it is hard to get the analytical forms of their phase terms and then to get the phase velocity by using these phase terms. Figure 1(b) shows the phase velocity as a function of x along the line ($y=w_0/2$, $z=w_0$) for continuous fundamental-mode laser beams with different beam widths $kw_0=60,30,10,6$, where the solid lines are for a paraxial approximation solution [12] [using Eqs. (2), (14), and (15)] and the dash-dotted lines are for a seventh-order correction solution [15] [using Eqs. (1) and (2) with numerical calculation]. Only when the laser beam is focused onto less than one wavelength ($kw_0=6$) do the differences become clear. Because the new formula is exactly derived from the wave equation, in principle, an accurate phase velocity of a real laser field can be obtained by using the new formula and the high-order correction solutions. The correction effect of the phase velocity is not evident in Fig. 1(b). Referring to Fig. 1(a), the phase velocity obtained via the phase term might be further away from the real cases.

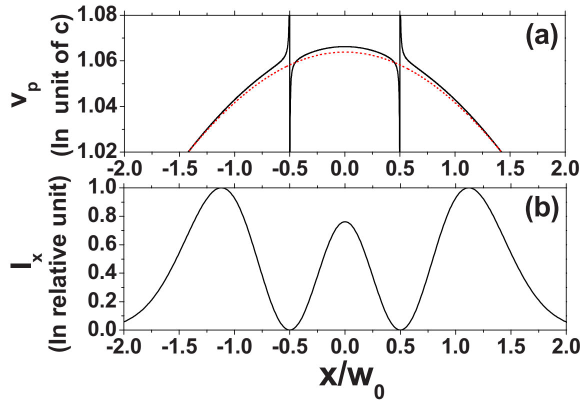

FIG. 2. (Color online) (a) The phase velocity as a function of x along the line $y=z=0$ (solid line using the new formula, dotted line using the old formula). (b) The intensity I_x as a function of x along the line $y=z=0$. A continuous HG(2,0)-mode laser beam with $w_0=10$ is used.

B. High-order mode case

According to Eq. (3), we can obtain the intensity contribution from the transverse electric component E_x,

$$I_x(\mathbf{r}) = \frac{a_{mn}^2 w_0^2}{w(z)^2}[H_m(X)H_n(Y)]^2 \exp\left(-\frac{2r^2}{w(z)^2}\right). \quad (17)$$

From Eq. (17), we find that the intensity of a high-order HG-mode laser beam alternates between zeros and peaks. Then, using the new formula, the phase velocity may show special points at the zeros where $\psi_{\mathbf{r}}(\mathbf{r})=0$. However, there is no such problem for the old formula. For an HG(m,0) mode laser beam, the special points may arise from the term $H_{m+2}(X)/H_m(X)$ in Eq. (15) at the zeros where $H_m(X)=0$. From Hermite polynomials, we find that $H_{m+2}(X)$ is exactly divisible by $H_m(X)$ when $m<2$. In order to verify these problems, here we consider the points in the line $y=z=0$ of an HG(2,0) mode laser beam

$$\frac{\nabla^2\psi_{\mathbf{r}}}{\psi_{\mathbf{r}}} = \frac{4x^2}{w_0^4} - \frac{12}{w_0^2} + \frac{1}{Z_R^2}\left[\left(\frac{2x^2}{w_0^2} - 3\right) - \frac{2w_0^2}{4x^2 - w_0^2}\right]. \quad (18)$$

Figure 2(a) shows the phase velocity as a function of x along the line $y=z=0$ (solid line using the new formula [Eqs. (2) and (18)], dotted line using the old formula). Here, a continuous HG(2,0) mode laser beam with $w_0=10$ is used. We find that the difference between the results obtained by these two approaches is very small except for two small ranges near the special points $x=\pm 0.5w_0$ which are exactly the points $H_2(X)=0$, corresponding to the intensity $I_x=0$. Figure 2(b) shows the intensity I_x as a function of x along the line $y=z=0$. Obviously, the phase velocity obtained with the new formula appears to have step changes near these two points. From the mathematical form, $\nabla^2\psi_{\mathbf{r}}/\psi_{\mathbf{r}} \to \infty$ when $I_x \to 0$ and then $v_p \to 0$. However, the extreme value of $\nabla^2\psi_{\mathbf{r}}/\psi_{\mathbf{r}}$ in this point is completely different for the different approximation directions, namely,

$$\lim_{|x|-w_0/2\to 0^-} \frac{\nabla^2\psi_{\mathbf{r}}}{\psi_{\mathbf{r}}} = +\infty, \quad (19)$$

$$\lim_{|x|-w_0/2\to 0^+} \frac{\nabla^2\psi_{\mathbf{r}}}{\psi_{\mathbf{r}}} = -\infty. \quad (20)$$

According to the new formula, Eq. (19) results in $v_p=0$. During this process of approximation, $\nabla^2\psi_{\mathbf{r}}/\psi_{\mathbf{r}}=0$ results in $v_p=c$. Then the phase velocity runs from c to zero rapidly. During the process of approximation in Eq. (20), $\nabla^2\psi_{\mathbf{r}}/\psi_{\mathbf{r}} \to -k^2$ results in $v_p \to \infty$ and the phase velocity will become an imaginary speed if $\nabla^2\psi_{\mathbf{r}}/\psi_{\mathbf{r}} < -k^2$. Here the imaginary speed appears not only at a point but also in small regions near the points $\psi_{\mathbf{r}}=0$. There is no such problem of imaginary speed for the old formula. The phase jump may not be strange near a zero point of amplitude, but not all of the points where $I_x=0$ will be such discontinuities for the phase velocity. It depends on the nearby distribution of the amplitude. For example, $\psi_{\mathbf{r}=0}=0$ for the HG(1,0)-mode, but $\lim_{r\to 0^-}(\nabla^2\psi_{\mathbf{r}}/\psi_{\mathbf{r}}) = \lim_{r\to 0^+}(\nabla^2\psi_{\mathbf{r}}/\psi_{\mathbf{r}})$ and this point is a continuous point for phase velocity.

On the other hand, the intensity distribution of a high-order HG-mode laser beam does not have rotational symmetry with respect to the optical axis. Accordingly, the phase velocity distribution obtained by the new formula may not be rotationally symmetric. However, using the old formula [Eq. (11)], we can get a phase velocity distribution with a rotational symmetry. Because Eq. (14) is rotationally symmetric, the symmetry of the phase velocity distribution obtained with the new formula may be broken by the term given in Eq. (15). If the Hermite polynomials can satisfy the following relation:

$$\frac{H_{m+2}(X)}{H_m(X)} = 2(2X^2 - 2m - 1), \quad (21)$$

Eq. (15) and then the phase velocity distribution obtained by the new formula will be rotationally symmetric. It is easy to test that $m=0$ satisfies the above relation, as this accords with the symmetric intensity distribution of a fundamental [HG(0,0)] mode. It is interesting that $m=1$ also satisfies the above relation even though the intensity distribution of the HG(1,0) mode is not rotationally symmetric. We do not exclude that an $m>1$ may exist which can satisfy this relation. Here we find that $m=2$ does not satisfy this relation. This character is different from that of the old formula.

IV. MEASUREMENT PROBLEM

When we use the above new formula to measure the phase velocity distribution of a real optical field (nonplanar wave), it is very difficult to disjoin the intensity contributed from one component and that from another. An example of this problem is measuring the phase velocity E_x of an x-polarized laser beam ($E_y=0$). According to *Maxwell*'s equations, the electric component E_z can be approximately obtained by using $E_z=(i/k)(\partial E_x/\partial x)$. The intensity we measured is $I=|E_x|^2+|E_z|^2$, but the intensity we wanted is $I_x=|E_x|^2$. If we use the measured intensity to calculate the phase velocity of E_x, the systematic error introduced by the component E_z is

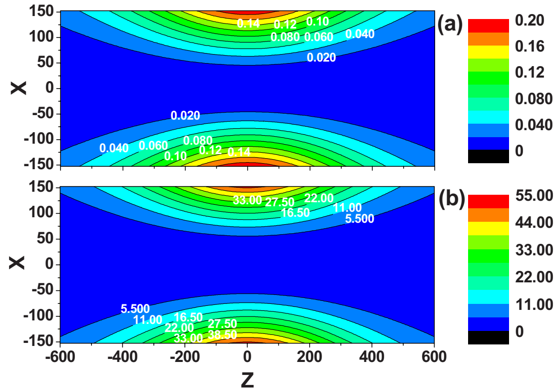

FIG. 3. (Color online) (a) Systematic errors scaled by $|\Delta v_p| \times 10^3$ in the plane $y=0$. (b) The distribution of phase velocity scaled by $|v_p - c| \times 10^3$ in the plane $y=0$. A continuous fundamental-mode Gaussian laser beam with beam width $w_0=30$ is used.

$$\Delta v_p = \frac{c}{\sqrt{1 + \frac{1}{2k^2}\left(\frac{\nabla^2 I}{I} - \frac{1}{2}\frac{(\nabla I)^2}{I^2}\right)}} - \frac{c}{\sqrt{1 + \frac{1}{2k^2}\left(\frac{\nabla^2 I_x}{I_x} - \frac{1}{2}\frac{(\nabla I_x)^2}{I_x^2}\right)}}. \tag{22}$$

Figure 3 shows the contour of (a) the above systematic error scaled by $|\Delta v_p| \times 10^3$ and (b) the phase velocity of E_x scaled by $|v_p - c| \times 10^3$ in the $y=0$ plane, where a continuous fundamental-mode Gaussian laser beam with a beam width $w_0=30$ is used. From the distribution of E_z [Fig. 1(a) in Ref. [11]], we find that the regions with a biggish error do not overlap the regions with a higher distribution of E_z even though the error is introduced by E_z. This is especially the case in the central region where E_z is almost zero, but the error is nonzero. This is because the phase velocity is related to the second-order differential coefficient of $|E_z|$. From the distribution of phase velocity shown in Fig. 3(b), we find that the error will increase as the phase velocity deviates from the speed of light c. As w_0 increases, the rate of change of ∇E_z will decrease and the error will also decrease correspondingly. If $w_0 \to \infty$, we can get $\Delta v_p \to 0$ because $E_z \equiv 0$ for plane waves. Our calculation shows that the systematic error $|\Delta v_p|$ is a very small quantity. On the other hand, since the phase velocity is very close to the speed of light, the deviation $|v_p - c|$ between them is also a very small quantity, as shown in Fig. 3(b). In the regions where $|v_p - c| \approx 0$, we find that $|\Delta v_p|$ is even larger than $|v_p - c|$.

V. DISCUSSION AND CONCLUSION

In summary, the new formula for the phase velocity of electromagnetic waves is an interesting one and some aspects of it need attention. For a fundamental-mode Gaussian laser beam, the results obtained with this new formula and directly from the phase term are almost the same. For high-order Gaussian-mode laser beams, discontinuites sometimes appear in the distribution of the phase velocity obtained with the new formula. The value of the phase velocity appears to have step changes near these discontinuites, and an imaginary speed even appears. However, the phase velocity obtained with the old formula is always continuous. In addition, for some high-order modes, the distribution obtained with the new formula is not rotationally symmetric. This is different from the results from the old formula. The agreement between these two approaches is reasonable because the Gaussian laser beams satisfy the *Helmholtz* equation under the SVEA and the new formula is exactly derived from the *Helmholtz* equation. However, their differences for high-order modes are not a problem of quantity because they still appear even for a large focused spot size w_0. The real reason behind this is not yet clear and needs further study. When we use the new formula to calculate the phase velocity of a wave field component by measuring the field intensity, we should take account of the fact that the other components of the wave field always introduce errors.

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